

Transactions Briefs

Frequency Limitations in Circuits Composed of Linear Devices

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Abstract—This paper investigates limitations on the frequency response of networks constructed out of components specified by their small signal models. Tellegen's theorem is used to find tight bounds on the maximum frequency of oscillation. The problem is reduced to deciding whether zero is in the numerical range of a complex non-Hermitian matrix. A decision method is presented, and transistor and negative resistance amplifier examples are developed.

I. INTRODUCTION

The frequency limitations of a circuit are determined by the frequency characteristics of its components and the cleverness with which those components are interconnected. It is possible to discover bounds on the frequency response of a circuit based only on the characteristics of the components. For instance, it is known that the natural frequencies (poles and zeros of the admittance) of any one-port constructed only of linear passive resistors and capacitors must lie on the negative real axis of the s -plane ($\omega = 0$ and $\sigma \leq 0$, where $s \triangleq \sigma + j\omega$) [1].

In an example of more relevance today, we can examine a circuit composed of incrementally active devices, such as transistors, and ask, what is the range of natural frequencies achievable by linear time-invariant autonomous circuits built of these components? Fig. 1(a) illustrates a simple model for a MOS transistor together, in Fig. 1(b), with the permitted natural frequencies of networks constructed solely of these devices. No such network, regardless how cleverly constructed, can have pure sinusoidal natural frequencies of oscillation above ω_{\max} . This example was adapted from the work of Thornton [2], [3]. In Fig. 1(b), $R = 0$ and the permitted and forbidden regions are separated by the line

$$\sigma = \frac{g_m^2}{4g_D C_{GS}} - \frac{\omega^2 g_D C_{GS}}{g_m^2}. \quad (1)$$

The reason poles can lie on the $j\omega$ axis without inductors is that the transistors can be used to make gyrators which enable capacitors to emulate inductors. Note how the permitted natural frequencies in Fig. 1(b) reduce to that of an RC network as g_m goes to zero.

The natural frequencies of a transistor are closely related to the frequencies at which that transistor can be active. Mason was the first to develop a figure of merit for a transistor in a lossless reciprocal embedding with his work on the *unilateral gain* of a linear two port [4]. Later, Thornton published his work on the allowed natural frequencies of active RC networks [5], relating it

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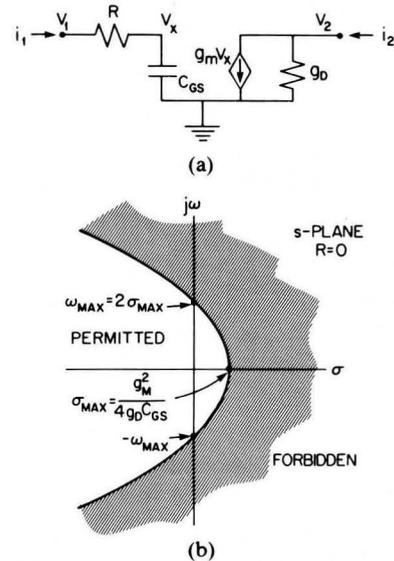


Fig. 1. Simple small signal model of a MOS transistor (a) and the permitted and forbidden natural frequencies (b) of networks built out of resistors, capacitors, and these transistors ($R = 0$).

back to Mason's. Kuh and Desoer generalized both works by expanding the class of networks considered and allowing nonreciprocal embeddings [6], [7]. In this paper, the class of networks considered will be further expanded, systematized, and extended in a way that permits a simple computation to test a complex frequency to determine whether or not it can be a natural frequency of a network composed of any interconnection of components specified by their small signal models. A general statement of the problem is:

Given a set of linear components, taken in any multiplicity and impedance scaled by any positive real number, what is the lowest sinusoidal frequency, or more generally the largest region in the s -plane, for which one can prove that connected networks built of these components cannot have natural frequencies?

The first step is to reformulate the problem in terms of Tellegen's theorem [8] to achieve increased generality. This makes it possible to express the problem in terms of any set of components \mathcal{S} characterized by their small signal models and interconnected by wire. This generalization is important because the performance of most modern devices is dominated by a complex network of parasitics. To neglect these parasitics is to be overly optimistic, such as in the case of Fig. 1 where the use of inductors can increase ω_{\max} to ∞ . The reason ω_{\max} can be increased without bound in this simple model is that all of the internal capacitors are directly connected across external terminals and hence can be resonated out, at any desired frequency, with a parallel inductor. This clearly nonphysical prediction occurs because the model used was excessively simple. Note that σ_{\max} does not increase because both inductors and capacitors absorb power for all time when excited with a growing exponential voltage or current waveform (in this problem $\omega = 0$ at σ_{\max}).

The Tellegen formulation leads to a complex non-Hermitian matrix $Y(s)$ which captures the relevant frequency-domain information. In this matrix, each type of component need be represented only once. A simple test on $Y(s)$ will then be presented which can discover whether or not nonzero voltage and current solutions are permitted at a frequency s . From this we can discover regions of the s -plane where, independent of the circuit topology, the network has no natural frequencies and cannot oscillate. In Section IV we will show that these bounds are, in fact, tight.

We have motivated this investigation with the goal of mapping out the s -plane into forbidden and permitted regions. In the next three sections we will focus the discussion on the subproblem of examining a specific point in the plane, s_0 , and discovering whether or not it is in the forbidden region. In Section V (Examples) we again look at the s -plane as a whole but, instead of mapping it completely, we will parameterize it in terms of two points where the line separating the forbidden and permitted regions intersect the real and imaginary axes: σ_{\max} and ω_{\max} .

II. TELLEGEN FORMULATION AND THE CONSERVATION OF COMPLEX POWER

One of the many special cases of Tellegen's theorem states that the inner product of branch voltages v and branch currents i is zero, where v is the column vector of Laplace transforms of branch voltages and i is the column vector of Laplace transforms of branch currents, with associated reference directions imposed. The branches are numbered. We have

$$0 = v^H(s) i(s) \quad (2)$$

where v^H denotes the complex conjugate transpose of v . Equation (2) can be interpreted as the conservation of complex power. The fact that two real quantities are conserved in (2)—the physically intuitive real power and the more enigmatic imaginary power—makes this problem somewhat nonstandard.

Tellegen's theorem for a network can also be expressed in terms of the port or terminal voltages and currents of its subnetworks [9]. Let $\mathcal{S} \triangleq \{\mathcal{N}_1, \dots, \mathcal{N}_M\}$ be any set of linear multiports, not necessarily all of the same size, characterized by an associated set of admittance matrices $\mathcal{Y} \triangleq \{Y_1(s), \dots, Y_M(s)\}$. For multiport number k we have

$$i_k = Y_k v_k. \quad (3)$$

Let \mathcal{M} be any network obtained by producing a connected network from the elements of \mathcal{S} and ideal wire, without violating the port assumptions of the admittance matrices. Tellegen's theorem for \mathcal{M} states that any solution of the network must satisfy

$$0 = \sum_{k=1}^M v_k^H i_k \quad (4)$$

or

$$0 = \sum_k v_k^H Y_k v_k. \quad (5)$$

Equation (5) is a quadratic form but note that the admittance matrices Y_k are typically non-Hermitian.

In deference to the leverage of integrated circuit technology, it is greatly desirable to generalize the set of networks considered by the theory to networks which include, not just one, but any number of instantiations of the components from the set \mathcal{S} . Limitations on the frequency behavior of these networks can be

discovered by investigating the natural frequencies at which Tellegen's theorem permits nontrivial voltage solutions.

We begin by defining a natural frequency. See, for instance, the discussion in [21].

Definition 1

For a linear network, described in the frequency domain, with all independent sources set to zero, a *network solution at the complex frequency s_0* is a set of complex branch voltages and currents that satisfy Kirchhoff's voltage and current laws, and the element constitutive relations, where the element impedances and admittances are evaluated at s_0 . A *natural frequency* of the network is a value of s_0 at which a nonzero network solution exists. The *maximum frequency of oscillation* ω_{\max} of the network is the maximum sinusoidal natural frequency $\text{Im}(s_0)$ for which the natural frequency s_0 has zero real part.

Since ω_{\max} is a natural frequency, a nonzero network solution exists at $s_0 = j\omega_{\max}$. In this work, we are interested in the allowed natural frequencies of not just a specific network, but rather the possible natural frequencies of large collections of elements taken from the set \mathcal{S} . The next two definitions clarify these concepts.

Definition 2

Let $\mathcal{G} \triangleq \{\mathcal{L}_1, \dots, \mathcal{L}_N\}$ be any set of multiports characterized by admittance matrices that are positive scalar multiples of matrices from the set \mathcal{Y} ; i.e., for each $i \in \{1, \dots, M\}$ the admittance matrix of \mathcal{L}_i is of the form $a_i Y_{K(i)}(s)$, where $a_i \in R$ is positive and where $K(\cdot): \{1, \dots, N\} \rightarrow \{1, \dots, M\}$. Let \mathcal{N} be any connected network obtained by interconnecting $\mathcal{L}_1, \dots, \mathcal{L}_N$ using only ideal (multiwinding) transformers and ideal connecting wire. Tellegen's theorem implies that any solution $v_1(s), \dots, v_N(s)$ to the network \mathcal{N} satisfies

$$0 = \sum_i^N a_i v_i^H(s) Y_{K(i)}(s) v_i(s). \quad (6)$$

A frequency $s \in C$ is not a natural frequency of \mathcal{N} if $(v_1^T(s), \dots, v_N^T(s)) = 0$ is the unique solution to (6). A frequency s is *complex power forbidden* if no network \mathcal{N} , as specified above, can be constructed to have s as a natural frequency.

In other words, no such network \mathcal{N} can be constructed to have a natural frequency $s_0 \in C$ unless there is a nonzero voltage solution to (6) at $s = s_0$. Using this criterion we can divide the s -plane into forbidden and permitted regions; see Fig. 1(b).

Definition 3

A frequency $s \in C$ is *complex power permitted* if it is not complex power forbidden for some \mathcal{N} as specified in Definition 2.

In other words, a frequency is complex power permitted if the special case of Tellegen's theorem stated in (6) does not rule it out as a natural frequency for all such \mathcal{N} . We will show in Section IV that if a frequency is complex power permitted for some \mathcal{N} then there exists an \mathcal{N} with that natural frequency.

III. MAIN RESULT

Equation (6) is a quadratic form. These forms have been widely studied in the mathematics literature. Central to much of this study is a quantity called the *numerical range*. We will use the numerical range concept extensively in the remainder of this paper. We present below the definition of the numerical range of a complex matrix. A brief listing of some of its well-known properties are given in the Appendix. The numerical range of $B \in C^{n \times n}$ is also called the *field of values* or *Wertvorrat* of B .

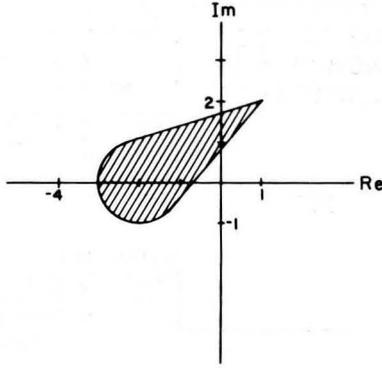


Fig. 2. The numerical range of the non-Hermitian 3×3 matrix B given in the text. Neither the Hermitian nor anti-Hermitian parts of B are definite, yet $0 \notin W(B)$.

Definition 4

The numerical range of a complex matrix $B \in C^{n \times n}$ is the set of $W(B)$ of all complex numbers of the form $x^H B x$, where x varies over all vectors on the unit sphere, $x^H x = 1$ [11]–[15].

Most remarkably, the numerical range of a matrix is always a convex, bounded, and closed (Fact 1, in the Appendix). Fig. 2 illustrates the numerical range of the matrix B :

$$B = \begin{pmatrix} -2 & 0 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 1+2j \end{pmatrix}.$$

Only one of its eigenvalues (see Facts 2 and 6) is on the boundary.

It is desirable to have a simple test on the set of admittance matrices \mathcal{Y} which tells whether a frequency s is complex power forbidden or permitted. Theorem 1, below, is the basis for such tests.

Theorem 1 (Main Result)

Let $\mathcal{Y} \triangleq \{Y_1(s), \dots, Y_M(s)\}$ be any finite set of admittance matrices (not necessarily of the same size). Let $\{\mathcal{L}_1, \dots, \mathcal{L}_N\}$ be any set of linear multiports characterized by admittance matrices that are positive scalar multiples of matrices from the set \mathcal{Y} ; i.e., for each $i \in \{1, \dots, N\}$ the admittance matrix \mathcal{L}_i is of the form $a_i Y_{K(i)}(s)$, where $a_i \in R$ is positive and where $K(\cdot): \{1, \dots, N\} \rightarrow \{1, \dots, M\}$. Define

$$Y(s) \triangleq \text{diag} \{ Y_1(s), \dots, Y_M(s) \}. \quad (7)$$

Let \mathcal{N} be any connected network obtained by interconnecting $\mathcal{L}_1, \dots, \mathcal{L}_N$ using only ideal (multiwinding) transformers and ideal connecting wire. For $s_0 \in C$, a complex frequency not a pole of $Y(s)$, s_0 is complex power permitted iff $0 \in W(Y(s_0))$. Conversely, s_0 is complex power forbidden iff $0 \notin W(Y(s_0))$.

Note that Theorem 1 suffices to demonstrate that s_0 is not a natural frequency of a remarkably large class of networks: the class of all networks made of interconnections of any number of multiports described by admittance matrices in the set \mathcal{Y} , or positive scalar multiples of admittance matrices in \mathcal{Y} . There is no requirement that $N \leq M$, i.e., the number of elements in the networks is unlimited.

Proof of Theorem 1:

With a change of coordinates, (6) becomes

$$0 = \sum_i^N x_i^H(s) Y_{K(i)}(s) x_i(s) \quad (8)$$

where

$$x_i(s) \triangleq \sqrt{a_i} v_i(s). \quad (9)$$

Let $x^T \triangleq (x_1^T, \dots, x_N^T)$, let $\|x\|^2$ denote the L_2 inner product $x^H x$, and let \mathcal{S}_C denote the unit sphere in C^n , i.e., $\mathcal{S}_C \triangleq \{x \in C^n \mid \|x\| = 1\}$. Clearly (8) has a nonzero solution \hat{x} iff (8) has a solution $x \triangleq \hat{x} / \|\hat{x}\|$ lying in \mathcal{S}_C . Since $x(s) = 0$ is always a solution to (8), it is unique iff $0 \notin W(\text{diag}(Y_{K(1)}(s), \dots, Y_{K(N)}(s)))$. Using Fact 11, $W(\text{diag}(Y_{K(1)}(s), \dots, Y_{K(N)}(s))) \subseteq W(\text{diag}(Y_1(s), \dots, Y_M(s)))$. Thus $0 \in W(\text{diag}(Y_{K(1)}(s), \dots, Y_{K(N)}(s))) \Rightarrow 0 \in W(Y)$. Equality occurs when every element in Y occurs at least once in the series $Y_{K(1)}(s), \dots, Y_{K(N)}(s)$; i.e., every element of S occurs in N .

Therefore $0 \in W(\text{diag}(Y_{K(1)}(s), \dots, Y_{K(N)}(s)))$ for some \mathcal{N} , as specified in Theorem 1, iff $0 \in W(\text{diag}(Y_1(s), \dots, Y_M(s)))$. Thus the frequency s is complex power permitted iff $0 \in W(Y(s))$ and complex power forbidden iff $0 \notin W(Y(s))$. ■

Johnson [17] and Ballantine [18] present iterative and noniterative decision methods, respectively, for determining whether $0 \in W(B)$ for a given matrix B . A method which we have used for the examples in Section V is based on Theorem 2 below. This method is a cross between that of Johnson and the one used in Appendix I of Thornton [5]. It differs from Johnson in that eigenvalues and eigenvectors need not be computed but it lacks the convenient mechanism for error control which Johnson's technique intrinsically provides. It is also similar to the method Thornton used to decide if $0 \in W(G + sC)$ where C is normal and nonsingular. Because C is normal it can be diagonalized, and, with a change of coordinates, his problem becomes one of deciding whether $0 \in W(B + sI)$. By Fact 5, this is equivalent to asking whether $-s \in W(B)$. Thornton, who did not use numerical range concepts, actually solves the question of $-s \in \text{Co}\{W(B)\}$, the convex hull of $W(B)$, but by Fact 1, $\text{Co}\{W(B)\} = W(B)$ so his results are actually stronger than he shows.

If Y were Hermitian (or even anti-Hermitian) then testing to see if $0 \in W(Y)$ would be straightforward. If Y is either positive- or negative-definite then no nontrivial solutions exist. Standard tests for definiteness include looking at the pivots, eigenvalues, or subdeterminates of Y [10].

We can examine necessary conditions for (8) to have a nonzero solution by looking separately at the Hermitian and anti-Hermitian parts of (8). Let

$$Y_H \triangleq \frac{1}{2}(Y + Y^H) \quad (10)$$

and

$$Y_{AH} \triangleq \frac{1}{2}(Y - Y^H) \quad (11)$$

where Y_H is the Hermitian part of Y and Y_{AH} is the anti-Hermitian part. Testing these parts separately, however, generates an overly optimistic prediction of the size of the permitted region because, for instance, when looking at Y_H we are allowing $v^H Y_{AH} v$ to take on any (imaginary) value. Said another way, if \mathcal{V}_h is the solution set to $0 = v_h^H Y_H v_h$ and \mathcal{V}_{ah} , the solution set to $0 = v_{ah}^H Y_{AH} v_{ah}$, the solution set to (8) is the intersection of \mathcal{V}_h and \mathcal{V}_{ah} , which can easily consist of

only the origin even when \mathcal{V}_h and \mathcal{V}_{ah} are quite large. In (8) we are looking for simultaneous solutions to two quadratic forms.

While definite tests on the Hermitian part of Y is not quite what we want to do, it is close. The following theorem is the basis for a simple method of computing the boundary separating the forbidden and permitted regions.

Theorem 2 (Decision Method)

For $Y(s_0) \in C^{n \times n}$, $0 \notin W(Y(s_0))$ iff

$$A(\theta, s_0) \triangleq \frac{1}{2}(Y(s_0) e^{j\theta} + Y^H(s_0) e^{-j\theta}) \quad (12)$$

is positive-definite for some $\theta \in [0, 2\pi)$.

Said another way, using the structure of Fact 10, if $Y_k(s_0)e^{j\theta} + Y_k^H(s_0)e^{-j\theta}$, the Hermitian part of each phase-shifted subcomponent admittance matrix, is positive definite for all $Y_k(s_0) \in \mathcal{Y}$ at the same θ then no network made from the elements of \mathcal{Y} , taken in any multiplicity and admittance scaled by any nonnegative real number, is complex power permitted at $s_0 \in C$. The proof of Theorem 2 rests on Lemma 1 below.

Lemma 1

A closed convex set \mathcal{F} in the complex plane does not contain the origin iff there exists a rotation about the origin that carries all of \mathcal{F} into the open right-half plane.

Discussion of Lemma 1:

This lemma is obvious but the interested reader could construct a proof by noting that two closed convex sets, in our case \mathcal{F} and the origin, containing no points in common must lie on opposite sides of a separating hyperplane (in this case a straight line) [16]. To show that a rotation about the origin carries all of \mathcal{F} into the open right-half plane it is sufficient to show that there exists a rotation which brings the line into the open right half plane, parallel to the imaginary axis, with the origin (which has not moved) on one side and \mathcal{F} on the other.

Proof of Theorem 2:

By Fact 1, $W(Y(s_0))$ is convex and therefore by Lemma 1, $0 \notin W(Y(s_0))$ iff there exists a rotation $\theta \in [0, 2\pi)$ such that $e^{j\theta}W(Y(s_0))$ lies entirely in open right-half plane. Since $e^{j\theta}W(Y(s_0)) = W(e^{j\theta}Y(s_0))$ by Fact 3, there exists a $\theta \in [0, 2\pi)$ such that $W(e^{j\theta}Y(s_0))$ lies entirely in open right-half plane iff $0 \notin W(Y(s_0))$. By Fact 14, the projection of $W(e^{j\theta}Y(s_0))$ onto the real axis is $W(A(\theta, s_0))$. Thus $0 \notin W(Y(s_0))$ iff $W(A, \theta)$ is contained in the strictly positive reals, i.e., iff $A(\theta, s_0)$ is positive definite for some $\theta \in [0, 2\pi)$. ■

When solutions to (6) are found at the same s that Y is singular, further inspection of the result is required. When Y is singular, voltage solutions v can be sustained with zero current. These solutions, in which no real or reactive power flow through the network, are generally of little interest from the standpoint of this work. For the same reason, when formulating Y for multi-terminal elements, the definite form of the admittance matrix must be used. (Because the indefinite form of an admittance matrix is singular, zero is always in the numerical range by Fact 15. This just means that a constant can be added to the voltage on every node.)

IV. THE TIGHTNESS OF THE BOUNDS

In this section it is shown that the bounds derived in the last section are tight. In other words, we show that if a natural frequency is complex power permitted for a collection of devices then there exists a network constructed out of those devices and ideal transformers which has such a natural frequency. (We do not know whether such a network can be constructed without ideal transformers.) Our approach is to first note that if a natural frequency s_0 is complex-power permitted then there exists a nonzero voltage vector $v(s_0)$ which satisfies

$$0 = v^H(s_0)Y(s_0)v(s_0) \quad (13)$$

and an orthogonal current vector $i(s_0)$ which satisfies

$$i(s_0) = Y(s_0)v(s_0). \quad (14)$$

We then use $v(s_0)$ to synthesize a network which satisfies Kirchhoff's voltage and current laws (KVL and KCL). This synthesis is most straightforward if the elements of \mathcal{S} are reciprocal, and we

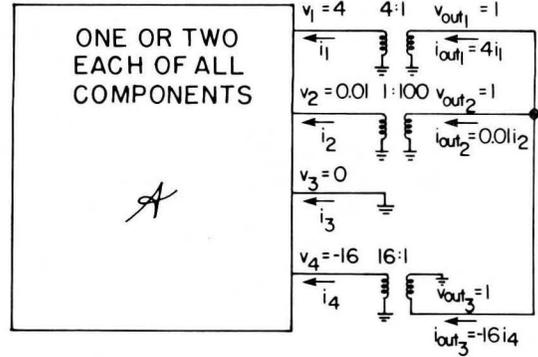


Fig. 3. Synthesis of embedding networks for reciprocal circuits, with some example numbers. When the voltages are transformed to unity, the currents satisfy KCL.

examine that case first:

Theorem 3 (Realizability for Reciprocal Devices)

For any natural frequency $s_0 \in C$ which is complex-power permitted by Theorem 1 and for which the elements of \mathcal{S} are reciprocal, a circuit exists that has s_0 as a natural frequency and is composed only of ideal wire, ideal transformers, and the elements of \mathcal{S} . Two each of the elements of \mathcal{S} may be required if $Y(s_0) \in C^{2 \times 2}$ and one each of the elements of \mathcal{S} are required otherwise.

This theorem makes the point that for networks composed of linear reciprocal devices, the bounds in Theorems 1 and 2 are tight; e.g., either a candidate natural frequency is complex-power forbidden or a relatively simple network can be built to realize it. There is no frequency regime wherein a natural frequency is complex-power permitted but unrealizable.

To prove Theorem 3 we need to introduce a second quadratic quantity called the *real-restricted numerical range*.

Definition 5

The *real-restricted numerical range* of a complex matrix $B \in C^{n \times n}$ is the set $R(B)$ of all complex numbers of the form $x^T B x$ where $x \in R^n$ varies over all vectors on the unit sphere $x^T x = 1$.

Clearly $R(B) \subseteq W(B)$. Other facts about the real-restricted numerical range are given in the Appendix. The fact that the real-restricted numerical range is closely related to the numerical range for symmetric matrices (Facts 17 and 18) enables us to find voltage vectors which satisfy conservation of complex power (i.e., equation (13)) and that have their phase angles aligned. This is used in the constructive proof below.

Proof of Theorem 3:

Because s_0 is complex-power permitted, $0 \in W(Y(s_0))$, and because the elements of \mathcal{S} are reciprocal, $Y(s_0)$ is symmetric. Let \mathcal{S}_R denote the unit sphere in R^n , i.e., $\mathcal{S}_R \triangleq \{x \in R^n | x^T x = 1\}$. The proof is best divided into two cases.

Case $n \neq 2$: By Fact 18, if there exists a complex voltage vector $\hat{v}(s_0) \in \mathcal{S}_C$ such that $0 = \hat{v}^H(s_0)Y(s_0)\hat{v}(s_0)$ then there also exists a real voltage vector $v(s_0) \in \mathcal{S}_R$ such that $0 = v^T(s_0)Y(s_0)v(s_0)$. By using one copy of the elements from \mathcal{S} a device \mathcal{A} can be constructed that has an admittance matrix $Y(s_0)$ as shown in Fig. 3. For $n \neq 2$, $0 \in W(Y(s_0))$ implies $0 \in R(Y(s_0))$.

Thus, we have shown that by using one copy of the elements of \mathcal{Y} , the voltage vectors can be chosen to be real. The current vectors may still be complex. The second step is to short to ground all terminals i for which the voltage v_i is zero. Next scale, with ideal transformers, all the nonzero voltages to 1, as illustrated in Fig. 3. Note that negative real voltages can be transformed to positive voltages by appropriately switching the transformer windings.

The synthesis is completed by trying all of the 1-V terminals together. The transformer output currents and voltages are orthogonal (i.e., $\mathbf{v}_{\text{out}}^H(s_0)\mathbf{i}_{\text{out}}(s_0) = 0$) since the input voltages and currents are orthogonal by assumption and ideal transformers conserve complex power. Because of this, and because $\mathbf{v}_{\text{out}}^H(s_0) = (1, \dots, 1)$, it is seen that the construction in Fig. 3 automatically satisfies KCL because $\mathbf{v}_{\text{out}}^H(s_0)\mathbf{i}_{\text{out}}(s_0) = \sum_k \mathbf{i}_{\text{out}k}(s_0) = 0$. This ends the proof for $n \neq 2$.

Case $n = 2$: For $n = 2$, the real-restricted numerical range and the numerical range are not necessarily the same. By Fact 18, $R(\mathbf{Y}) = \partial W(\mathbf{Y})$ for $n = 2$. We define a new admittance matrix $\mathbf{Y}'(s_0)$ which contains two of each element of \mathcal{Y} . That is, $\mathbf{Y}'(s_0) \triangleq \text{diag}(\mathbf{Y}(s_0), \mathbf{Y}(s_0))$. Then, by the same arguments as in the $n \neq 2$ case, there exists $\mathbf{v}(s_0) \in \mathcal{S}_R$ such that $0 = \mathbf{v}^T(s_0)\mathbf{Y}'(s_0)\mathbf{v}(s_0)$. Note that in this case $\mathbf{v}(s_0) \in \mathbb{R}^4$ while $\mathbf{Y}(s_0) \in \mathbb{C}^{2 \times 2}$. The remainder of the proof follows exactly as in the $n \neq 2$ case, except that two rather than one copy of the elements of \mathcal{S} are used. ■

The $n = 2$ case in the previous proof is somewhat disturbing but the reader can see, by counting degrees of freedom, that this case is bound to be problematic. There is only one degree of freedom in $\mathbf{x}^T \mathbf{x} = 1$, $\mathbf{x} \in \mathbb{R}^2$. Thus $\{\mathbf{x}^T \mathbf{B} \mathbf{x} | \mathbf{x}^T \mathbf{x} = 1, \mathbf{x} \in \mathbb{R}^2\}$ can be only a curve or a point in \mathbb{C} . By doubling the dimension of \mathbf{B} we dramatically increase the number of degrees of freedom. It is tempting to attribute the $n = 2$ "problem" to the nonphysical constraint of $\|\mathbf{v}(s_0)\| = 1$; however, even if this constraint is relaxed to $\|\mathbf{v}(s_0)\| > 0$, one still has difficulties for certain (symmetric) singular matrices such as

$$\mathbf{B} = \begin{pmatrix} 1 & j \\ j & -1 \end{pmatrix}.$$

Here, $W(\mathbf{B})$ is a circular disk, centered at the origin. Clearly, $0 \in W(\mathbf{B})$; nevertheless, $\mathbf{x}^T = (0, 0)$ is the only real vector \mathbf{x} which satisfies $0 = \mathbf{x}^T \mathbf{B} \mathbf{x}$, $\mathbf{x} \in \mathbb{R}^2$.

When some of the elements of \mathcal{S} are nonreciprocal, the voltage phasors cannot necessarily be aligned. We only know that $R(\mathbf{B}) \subseteq W(\mathbf{B})$ and, therefore, $0 \in W(\mathbf{Y}(s_0)) \Rightarrow 0 \in R(\mathbf{Y}(s_0))$. Nevertheless, it is still generally possible to find a synthesis given a nonzero voltage vector $\mathbf{v}(s_0) \in \mathbb{C}^n$ which satisfies (13).

Theorem 4 (Realizability for Nonreciprocal Devices)

For any natural frequency $s_0 \in \mathbb{C}$ which is complex power permitted by Theorem 1, a circuit exists that has s_0 as a natural frequency and is composed only of ideal wire, ideal transformers, and multiple copies of the elements of \mathcal{S} . No more than two each of the elements of \mathcal{S} are required.

Theorem 4 has the same structure as Theorem 3. That is, given the existence of complex voltage and current vectors which satisfy conservation of complex power, we use these vectors to synthesize a network based on these vectors. Unless it happens that a real voltage vector can be found, this synthesis will generally result in a more complicated network than the one in Fig. 3.¹

Proof of Theorem 4

As in Theorem 3, if $0 \in W(\mathbf{Y}(s_0))$ then there exists a nonzero $\mathbf{v}(s_0) \in \mathbb{C}^n$ such that

$$0 = \mathbf{v}(s_0)^H \mathbf{Y}(s_0) \mathbf{v}(s_0). \quad (15)$$

If it is also true that $0 \in R(\text{diag}\{\mathbf{Y}(s_0), \mathbf{Y}(s_0)\})$, then the synthesis in Theorem 3 is used. If not, then the following synthesis is appropriate.

¹The elegant synthesis shown in Fig. 3, which uses only two each of the components in \mathcal{S} is due in part to Professor Paul Penfield, Jr.

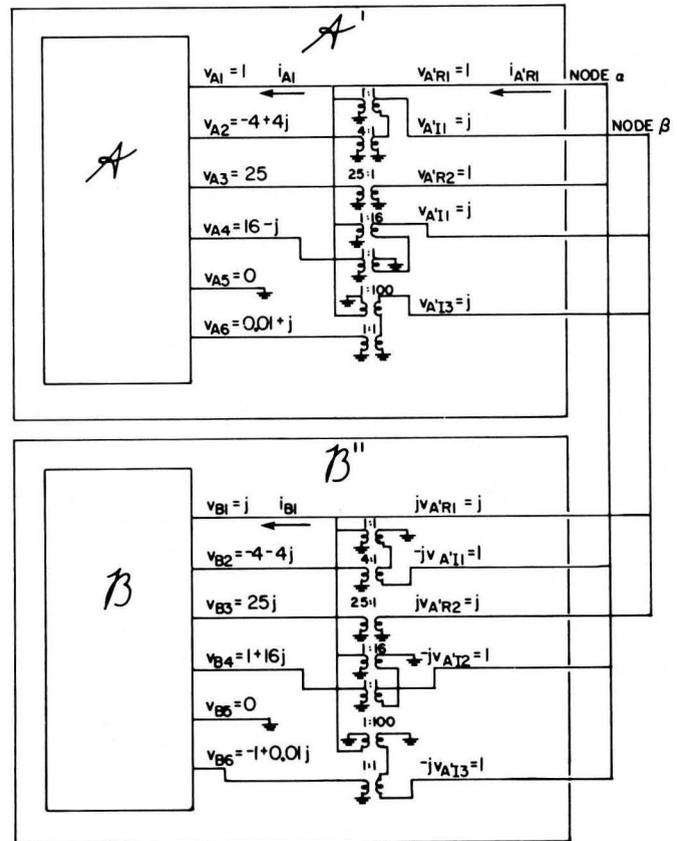


Fig. 4. Synthesis of embedding networks for nonreciprocal circuits, with some example numbers. when $0 \notin R(\mathbf{Y}(s_0))$, the voltage vector cannot be chosen real but it can be transformed into in-phase and quadrature components.

Step 1: Create the first copy \mathcal{A} of the network which includes one each of the elements of \mathcal{S} and is represented by the admittance matrix $\mathbf{Y}(s_0)$. (If v_1 , the first element of $\mathbf{v}(s_0)$, is zero, reorder the terminals of \mathcal{A} such that a nonzero voltages exist on terminal 1 of this network.) Define $\mathbf{v}_A(s_0) \triangleq \mathbf{v}(s_0)/v_1(s_0)$. This voltage vector is a solution to (13). Associate \mathbf{v}_A with the terminals of \mathcal{A} . Short to ground all terminals i of \mathcal{A} for which the voltage v_i is zero.

Step 2: We now consider only the terminals with nonzero voltage. As in Theorem 3, scale with ideal transformers all of the purely real voltage components of $\mathbf{v}_A(s_0)$ to 1. Call this vector of unit voltages, at the terminals of \mathcal{A}' , $\mathbf{v}_{A'R}$ and the associated current vector $\mathbf{i}_{A'R}$. Note that $\mathbf{i}_{A'R}$ is generally not real. With ideal transformers scale all the purely imaginary components of $\mathbf{v}_A(s_0)$ to j . For the remaining complex voltage components of $\mathbf{v}_A(s_0)$, use ideal transformers as illustrated in Fig. 4 to scale the imaginary component to j and subtract off the real component using a scaled version of the real voltage on terminal 1, v_1 . Call this network \mathcal{A}'' . Call the vector of imaginary voltages, at the terminals of \mathcal{A}'' , $\mathbf{v}_{A''I}$ and the associated complex current vector $\mathbf{i}_{A''I}$. Because ideal transformers preserve the assumed orthogonality of the voltages and currents,

$$\mathbf{v}_{A'R}^H \mathbf{i}_{A'R} + \mathbf{v}_{A''I}^H \mathbf{i}_{A''I} = \sum_m \mathbf{i}_{A'R_m} - j \sum_k \mathbf{i}_{A''I_k} = 0. \quad (16)$$

Step 3: Create a second copy \mathcal{B}' of the network built so far. For this second copy, multiply all voltages and currents by j . That is,

$$\mathbf{v}_B(s_0) = j \mathbf{v}_A(s_0). \quad (17)$$

The voltages on the terminals of \mathcal{B}' are either j or -1 .

Step 4: Use ideal transformers to invert the polarity of all real voltages which appear at the terminals of \mathcal{B}' . This will transform all of the -1 voltages to 1. In Fig. 4 this is done by switching the windings of already existing transformers. Call this new network \mathcal{B}'' .

Step 5: Connect together at node β all of the terminals of \mathcal{A}' and \mathcal{B}'' that have a potential of j . Connect together at node α all of the terminals of \mathcal{A}' and \mathcal{B}'' that have a potential of 1.

The currents leaving node α are $\sum_m i_{A'R_m} - j\sum_k i_{A'I_k}$ and the currents leaving node β sum to $j\sum_m i_{A'R_m} + \sum_k i_{A'I_k}$. Both of these sums are zero by (16). ■

The fact that two each of the circuit elements are generally needed for a successful synthesis seems to be fundamental. Some insight into this may be found by looking at another form of Tellegen's theorem. Tellegen's theorem tells us that, for any realizable network, not only is $v^H i = 0$, but $v^T i = 0$, as well. Ideal transformers and wire preserve $v^T i$; thus we know that (referring to Fig. 4) $v_A^T i_A + v_B^T i_B = 0$. We see that this works out quite trivially when v_A and v_B (and the associated currents) are phase-shifted by $\pi/2$, as specified in (17).

It is worth noting that the proofs of Theorems 3 and 4 do not, by themselves, enable a complete network synthesis. What is missing is the first step: given $0 \in W(Y(s_0))$, find a $v(s_0)$ which satisfies (13). Finding a satisfying $v(s_0)$ is straightforward when $0 \in \partial W(Y(s_0))$, by the use of Fact 9, but the situation is more involved in the general case and is beyond the scope of this paper.

V. EXAMPLES

In this section we will go through three examples to illustrate the scope and utility of the theory.

5.1. Negative Resistance Amplifier

The first example is a negative-resistance reflection amplifier constructed from three two-terminal elements: a negative resistance amplifier with a parallel parasitic capacitor and a series parasitic resistor; an inductor with a series parasitic resistor; and a resistor. The impedance of the amplifier is

$$Z_{\text{amp}}(s) = R_S + \frac{1}{sC - G}. \quad (18)$$

The inductor has a parasitic series resistance R_L , where $\omega_L \triangleq R_L/L$ is a constant of the technology. We have

$$Z_L(s) = R_L + sL = (\omega_L + s)L. \quad (19)$$

The passive resistor has a driving-point impedance $Z_R(s) = R_R$. The impedance matrix \mathbf{Z} for \mathcal{S} is

$$\mathbf{Z} = \text{diag}(Z_{\text{amp}}, Z_L, Z_R). \quad (20)$$

Since \mathbf{Z} is diagonal, the eigenvalues appear on the diagonal and the eigenvectors are orthogonal. Solving for $W(\mathbf{Z})$,

$$W(\mathbf{Z}) = Z_{\text{amp}}\|x_1\|^2 + Z_L\|x_2\|^2 + Z_R\|x_3\|^2 \quad (21)$$

where

$$\|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2 = \|x\|^2 = 1. \quad (22)$$

Here, x_1 , x_2 , and x_3 are the eigenvectors of \mathbf{Z} . Fig. 5 illustrates the numerical range of \mathbf{Z} . The dashed lines indicate how $Z_{\text{amp}}(j\omega)$ and $Z_L(j\omega)$ move with increasing ω ($\sigma = 0$). After some algebra, one finds

$$\omega_{\text{max}} = \frac{1}{C} \sqrt{\frac{G}{R_S} - G^2 - \frac{\omega_L C}{R_S}}. \quad (23)$$

This solution is found for $\omega = \omega_{\text{max}}$ at $a_{11}(\theta) = a_{22}(\theta) = 0$.

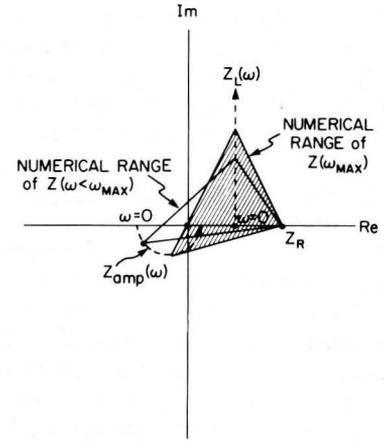


Fig. 5. The evolution of the numerical range of the reflection amplifier as ω increases.

TABLE I

element	realistic inductor	ideal inductor	units
R_S	0.1	0.1	Ω
G	1	1	Ω^{-1}
C	1	1	F
R_L	0.5	0	Ω
L	1	1	H
R_R	1	1	Ω
ω_L	0.5	0	s^{-1}
ω_{max}	2	3	s^{-1}

With $R_L = 0$, (23) reduces to the more optimistic expression for ω_{max} one would obtain by simply examining the Hermitian part of \mathbf{Z} . With the element values of Table I, $\omega_{\text{max}} = 2$ is predicted if complex power is conserved, but $\omega_{\text{max}} = 3$ is predicted if only real power is conserved (i.e., looking at only the Hermitian part of \mathbf{Z} or only $\theta = 0$).

The numerical range of $\mathbf{A}(\theta)$ is real; it is illustrated in Fig. 6 for $\omega = 2.5$. It lies between the minimum and maximum eigenvalues;

$$\lambda_{\min}(\mathbf{A}) = \min \text{Re} \begin{cases} Z_{\text{amp}} e^{j\theta} \\ Z_L e^{j\theta} \\ Z_R e^{j\theta} \end{cases} \quad (24)$$

and

$$\lambda_{\max}(\mathbf{A}) = \max \text{Re} \begin{cases} Z_{\text{amp}} e^{j\theta} \\ Z_L e^{j\theta} \\ Z_R e^{j\theta} \end{cases}. \quad (25)$$

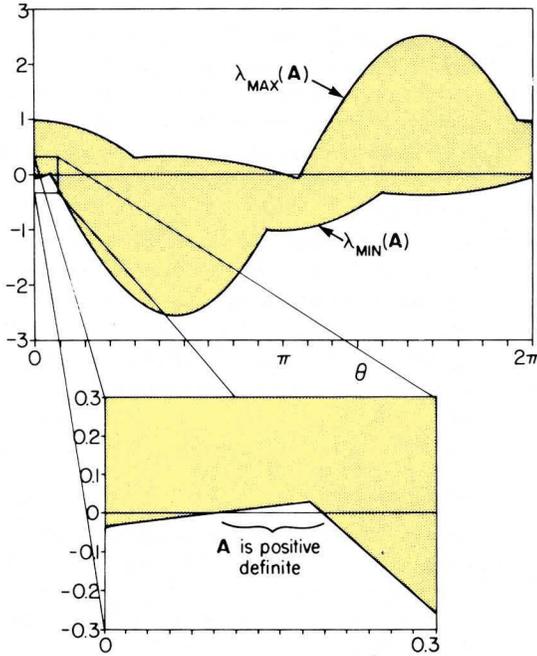
Note that $\lambda_{\min}(\mathbf{A}(\theta))$ can contain local maxima and points at which the derivative with respect to θ is undefined (see Fact 6).

5.2. Three-Terminal MOS Transistor

The second example we consider the interconnections of the three-terminal MOS transistor illustrated in Fig. 1(a). For this circuit,

$$\mathbf{Y} = \begin{pmatrix} \frac{sC}{sRC + 1} & 0 \\ \frac{g_m}{sRC + 1} & g_D \end{pmatrix}. \quad (26)$$

Note that the use of the source node as the datum does not imply


 Fig. 6. The numerical range of $A(\theta)$ at $\omega = 2.5$.

any restriction to common source configurations. A datum must be chosen so that a definite admittance matrix can be developed.

To aid our pursuit of closed-form solutions, we separate Y into Hermitian and anti-Hermitian parts. Rewriting $A(\theta, s)$,

$$A(\theta, s) = \cos \theta (Y_H(s) + jY_{AH}(s) \tan \theta). \quad (27)$$

For our purposes, the $\cos \theta$ scaling factor on $A(\theta)$ is inconsequential and may be neglected. Solutions for $X^H A x = 0$ at $\cos \theta = 0$ are generally not of interest because this would imply that the conservation of real power was unimportant. We define

$$A'(\delta, s) = \frac{1}{2} (Y_H(s) + j\delta Y_{AH}(s)) \quad (28)$$

where $\delta \triangleq \tan \theta$. For $s = \sigma$ we obtain

$$A'(\delta, \sigma) = \frac{1}{2} \begin{pmatrix} \frac{2\sigma C}{\sigma RC + 1} & (1 + j\delta) \frac{g_m}{\sigma RC + 1} \\ (1 - j\delta) \frac{g_m}{\sigma RC + 1} & 2g_D \end{pmatrix} \quad (29)$$

and for $s = j\omega$ we obtain

$$A'(\delta, \omega) = \frac{1}{2} \begin{pmatrix} \frac{2\omega^2 C^2 R + 2\omega C \delta}{1 + (\omega RC)^2} & (1 + j\delta) \frac{g_m}{1 - j\omega RC} \\ (1 - j\delta) \frac{g_m}{1 + j\omega RC} & 2g_D \end{pmatrix}. \quad (30)$$

Since $a'_{22} > 0$, the determinant test for positive definiteness only requires examining the $\det A'$. Note that this was not the case in the reflection amplifier example. $\det A'$ is quadratic in δ . Solving for $\det A' = 0$ (the frequency at which A' becomes positive definite) leads to

$$\sigma_{\max}(\delta) = \frac{-1 \pm \sqrt{1 + \frac{(1 + \delta^2) g_m^2 R}{g_D}}}{2RC} \quad (31)$$

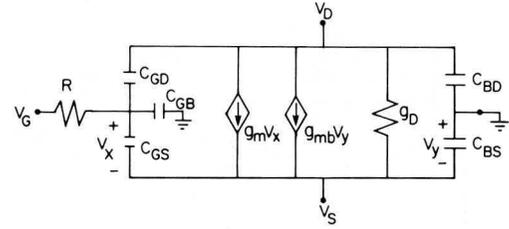


Fig. 7. A first-order MOS transistor model.

TABLE II

Parameter	Fig. 1a	Fig. 7	Trivial	Units
g_m	7×10^{-4}	7×10^{-4}	7×10^{-4}	Ω^{-1}
g_{mb}	0	2×10^{-5}	0	Ω^{-1}
C_{GS}	10^{-14}	10^{-14}	10^{-14}	F
C_{GD}	0	3×10^{-15}	0	F
C_{BS}	0	6×10^{-15}	0	F
C_{BD}	0	6×10^{-15}	0	F
C_{GB}	0	10^{-15}	0	F
g_D	8×10^{-5}	8×10^{-5}	0	Ω^{-1}
R	1000	1000	0	Ω
ω_{\max}	1.14×10^{11}	2.95×10^{10}	∞	s^{-1}
σ_{\max}	8.34×10^{10}	1.91×10^{10}	∞	s^{-1}
$f_{\max} = \omega_{\max}/2\pi$	18.1	4.69	∞	GHz
f_T	11.1	7.95	11.1	GHz

and

$$\omega_{\max}(\delta) = \frac{-\delta \pm \sqrt{\delta^2 + \frac{g_m^2 R}{g_D} (1 + \delta^2)}}{2RC} \quad (32)$$

Optimizing with respect to δ to find the min-max values for σ_{\max} and ω_{\max} , we obtain

$$\sigma_{\max} = \min_{\delta} \sigma_{\max}(\delta) = \frac{-1 + \sqrt{1 + \frac{g_m^2 R}{g_D}}}{2RC} \quad (33)$$

and

$$\omega_{\max} = \min_{\delta} \omega_{\max}(\delta) = \frac{g_m^2}{2g_D C} \left(1 + \frac{g_m^2 R}{g_D}\right)^{-1/2} \quad (34)$$

5.3. Four-Terminal MOS Transistor — A Numerical Example

The third example is an extension of the last. Here, realistic parasitics and a body terminal are added to the model of Fig. 1(a). The new model, which still does not take into account distributed effects or source and drain resistances, is shown in Fig. 7. Since the simple model of Fig. 1(a) is a special case of the model of Fig. 7, the two can be compared.

Table II gives the parameter values for the simplified (Fig. 1(a)) and full (Fig. 7) models. The parameter values are typical of MOSFETs found in VLSI circuits. The Fig. 7 values of σ_{\max} and ω_{\max} were obtained by computer. Note the large differences in speed predicted by the two models. For comparison, the transition frequencies f_T of the two cases are also given; f_T is the frequency at which the magnitude of the output short-circuit common-source current gain drops to 1. For these circuits, the f_T is

$$2\pi f_T = \frac{g_m}{C_{GS} + C_{GD} + C_{GB}} \quad (35)$$

Note, however, that the f_T of a transistor, while of proven usefulness, is not fundamental in the sense that ω_{\max} is fundamental, since it assumes a topology. This is why the series gate

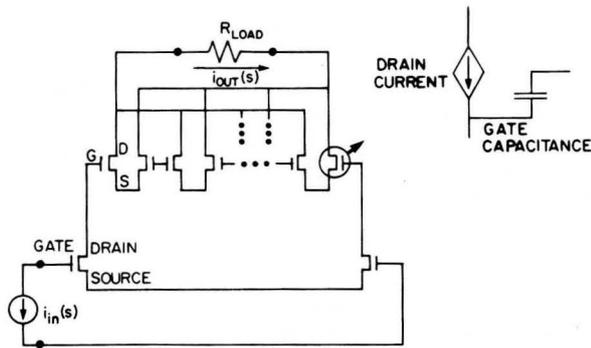


Fig. 8. A circuit to increase the effective f_T of a transistor. The trick to this circuit is that the capacitances are added in series while the currents are added in parallel.

resistor R does not appear in (35) despite its obvious importance to most practical circuits. The shortcomings of f_T are accentuated in the fourth column of Table II. While the f_T is finite, the maximum frequency of oscillation is infinite. A circuit which can realize current gain at any frequency is illustrated in Fig. 8. The trick to this circuit is that the gate capacitances are added in series while the drain currents are added in parallel. By using many transistors one can make the effective f_T as high as desired. Note that ω_{\max} , as defined in this paper, is more fundamental in the sense that a multiport composed of several identical transistors cannot have a ω_{\max} that exceeds that of the constituent transistors. In fact, ω_{\max} was deliberately defined in such a way as to ensure this property holds. The fourth column represents an exceptional case. More often, as in column three, the f_T is too optimistic a predictor of circuit performance.

In conclusion, a computational method for demonstrating that a frequency s_0 is not a natural frequency of the class of all networks made of interconnections of any number of multiports described by admittance matrices in the set \mathcal{Y} , or positive scalar multiples of admittance matrices in \mathcal{Y} , has been presented. The natural frequency bounds developed in this paper are tight. Since this technique is readily automated we believe it will be useful for finding the maximum frequency of oscillation of realistic, and hence complicated, device models. Since ω_{\max} includes the effects of all parasitics, we believe that it will find great utility in the comparison and development of competing technologies—e.g., GaAs MOSFET's and Si bipolar transistors. Equally important, not only can the effectiveness of active devices be investigated, but also the deleterious effects of imperfect interconnection elements, such as inductive resistors and resistive inductors, can also be included.

APPENDIX

Let \mathbf{B} be a complex matrix and $W(\mathbf{B}) \triangleq \{x^H \mathbf{B} x | x^H x = 1, x \in C^n\}$ be the numerical range of \mathbf{B} . Some facts [11]–[15], [17]–[19] about the numerical range are as follows.

- Fact 1:** The Toeplitz–Hausdorff theorem states the remarkable fact that the numerical range of a matrix is always convex, closed, and bounded. Note that $W(\mathbf{B}) \subset C$.
- Fact 2:** The spectrum (set of eigenvalues) of a matrix is contained in the numerical range, i.e., $\sigma(\mathbf{B}) \subseteq W(\mathbf{B})$.
- Fact 3:** For $s \in C$, $W(s\mathbf{B}) = sW(\mathbf{B})$.
- Fact 4:** For any unitary matrix \mathbf{U} , $W(\mathbf{U}^H \mathbf{B} \mathbf{U}) = W(\mathbf{B})$.
- Fact 5:** For $s \in C$, $W(\mathbf{B} + s\mathbf{I}) = W(\mathbf{B}) + s$ where \mathbf{I} is the identity matrix.

- Fact 6:** The boundary of $W(\mathbf{B})$, $\partial W(\mathbf{B})$, is a piecewise algebraic curve, and each point at which $\partial W(\mathbf{B})$ is not differentiable is an eigenvalue of \mathbf{B} .
- Fact 7:** If \mathbf{B} is normal, then $W(\mathbf{B}) = \text{Co}(\sigma(\mathbf{B}))$, where “Co” denote the closed convex hull of a set.
- Fact 8:** $W(\mathbf{B})$ is a segment of the real line iff \mathbf{B} is Hermitian. It is a segment of the imaginary line iff \mathbf{B} is anti-Hermitian.
- Fact 9:** Let $A(\theta) \triangleq \frac{1}{2}(\mathbf{B}e^{j\theta} + \mathbf{B}^H e^{-j\theta})$ for $\theta \in [0, 2\pi)$. Let $\lambda_{\max}(\theta) \in R$ be the maximum eigenvalue and $\mathbf{x}_{\max}(\theta)$ a corresponding eigenvector of $A(\theta)$ with $\|\mathbf{x}_{\max}(\theta)\| = 1$ (i.e., $A(\theta)\mathbf{x}_{\max}(\theta) = \lambda_{\max}(\theta)\mathbf{x}_{\max}(\theta)$), then $\mathbf{x}_{\max}^H(\theta)\mathbf{B}\mathbf{x}_{\max}(\theta) \in \partial W(\mathbf{B})$. Also $W(\mathbf{B}e^{j\theta}) \subset H(\lambda_{\max}(\theta))$ where $H(a)$, $a \in R$, is the half-plane defined by $\{z \in C | \text{Re } z \leq a\}$. The line defined by $\{z \in C | \text{Re } z = \lambda_{\max}(\theta)\}$ is a supporting hyperplane of $W(\mathbf{B}e^{j\theta})$ [15].
- Fact 10:** $W(\text{diag}(\mathbf{B}_1, \dots, \mathbf{B}_N)) = \text{Co}\{W(\mathbf{B}_1), \dots, W(\mathbf{B}_N)\}$ where “diag” denotes the construction of a block diagonal matrix; e.g., $W(\text{diag}(\mathbf{B}_1, \dots, \mathbf{B}_N)) = \sum_i x_i^H \mathbf{B}_i x_i$ where $\sum_i \|x_i\| = 1$ [15].
- Fact 11:** $W(\mathbf{B}_1) \subseteq W(\text{diag}(\mathbf{B}_1, \dots, \mathbf{B}_N))$. Trivially, $W(\mathbf{B}) = W(\text{diag}(\mathbf{B}, \mathbf{B}))$.
- Fact 12:** If \mathbf{B} is a 2×2 matrix then $W(\mathbf{B})$ is a closed elliptical disk with foci at the eigenvalues.
- Fact 13:** If $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$, then $W(\mathbf{B}) \subseteq W(\mathbf{B}_1) + W(\mathbf{B}_2)$. For example, if \mathbf{B}_H is the Hermitian part of \mathbf{B} and \mathbf{B}_{AH} is the anti-Hermitian part of \mathbf{B} , then \mathbf{B} is contained in the rectangular region defined by $\{z \in C | \text{Re } z \in W(\mathbf{B}_H), \text{Im } z \in W(\mathbf{B}_{AH})\}$ [19].
- Fact 14:** The projection of $W(\mathbf{B})$ onto the real axis is $W(\mathbf{B}_H)$ and its projection onto the imaginary axis is $W(\mathbf{B}_{AH})$.
- Fact 15:** If \mathbf{B} is singular, then $0 \in W(\mathbf{B})$.
- Fact 16:** $W(\mathbf{B}) = W(\mathbf{B}^T)$.

Let \mathbf{B} be a complex matrix and $R(\mathbf{B}) \triangleq \{x^H \mathbf{B} x | x^H x = 1, x \in R^n\}$ be the real-restricted numerical range of \mathbf{B} . Some facts about the real-restricted numerical ranges are

- Fact 17:** Let $\mathbf{B}_{\text{sym}} \triangleq (\mathbf{B} + \mathbf{B}^T)/2$ be the symmetric part of \mathbf{B} , then $R(\mathbf{B}) = R(\mathbf{B}_{\text{sym}})$ where $R(\mathbf{B})$ is the real-restricted numerical range.
- Fact 18:** If $\mathbf{B} \in C^{2 \times 2}$, then $R(\mathbf{B}) = \partial W(\mathbf{B}_{\text{sym}})$. If $\mathbf{B} \in C^{n \times n}$, $n \neq 2$, then $R(\mathbf{B}) = W(\mathbf{B}_{\text{sym}})$ [20].
- Fact 19:** $R(\mathbf{B}) \subseteq W(\mathbf{B})$.

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